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A NOTE ON OPTIMAL AUCTIONS

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Abstract

This paper considers a general optimal auction problem, with many goods and with a buyer's utility that can depend non-linearly in his type. We point out that incentive compatibility constraints may be binding even if virtual utilities are strictly increasing in the buyer's type. More importantly, optimal mechanisms may involve randomizations between different allocations.

1. Introduction

The literature on revenue maximizing mechanisms is not only central to auction theory, but to economic theory in general. In the seminal contributions of Myerson (1981) and Riley and Samuelson (1981) two key features appear: first, that revenue maximizing mechanisms are *deterministic*, in the sense that either a good is obtained with probability one by a buyer, or it is kept by the seller, and, second, that incentive constraints essentially disappear if the virtual utilities, (utility minus information rents), of obtaining a good are strictly monotonic in a buyer's type.

Our findings contrast both these fundamental features of those papers. We consider a general selling problem, with many goods and where a buyer's utility may depend non-linearly on his type. First we establish that incentive compatibility can be binding even if virtual utilities are strictly monotonic in type. Then we show that when the incentive constraints bind, optimal auctions may involve randomizations between different allocations.

When incentive constraints bind despite monotone virtual utilities, previous methods of solving for a revenue maximizing mechanism fail because they rely on making virtual utilities monotonic, ("ironing"). We propose a solution method and with its help show that an optimal mechanism may involve *randomizations* over different goods. In some sense, a certain range of consumer types will be offered a random "bundle" of goods. Our method has the advantage that it does not require mechanisms to be differentiable because it does not rely on standard variational methods, (for instance the use of Hamiltonians).[1] It can be also used when the designer is interested in efficiency maximizing, instead of revenue maximizing mechanisms. All the analysis goes though by replacing virtual utilities with actual utilities.

The fact that randomizations are a feature of revenue maximizing mechanisms can be viewed as quite surprising given that the buyers are risk neutral and types are single dimensional. In the continuum varieties model of Maskin and Riley (1989), the authors devote Section 5, to illustrate why randomizations will not be a feature of revenue maximizing auctions. Also Thanassoulis (2004) stresses that the randomizations in his environment are due to the fact that types are multi-dimensional. Why then do they appear in our environment? The reason is that we are considering a *finite* number of different products, and we allow for utilities to be *non-linear* in types. The feature of randomization is absent in the extreme cases of one and a continuum of identical goods, but it can appear in intermediate

[1] See for instance Lollivier and Rochet (1983).

cases where there is some discreteness in the number of goods and goods are heterogeneous.

2. The model

A risk neutral revenue-maximizer seller owns N indivisible, possibly heterogeneous, objects that are of 0 value to her and faces one risk neutral buyer, who maximizes expected surplus. The seller can bundle these N objects in any way she sees fit. An allocation $z \in Z$ is an assignment of objects to the buyer and to the seller. It is a vector with N components, where each component stands for an object and specifies who gets it. Therefore, the set of possible allocations is finite and given by $Z \subseteq 2^N$. The buyer's valuation from allocation z is denoted by $u^z(v)$ and it depends on a preference parameter v which is private information and is distributed on $V = [\underline{v}, \bar{v}]$, according to a distribution F that has a strictly positive and continuous density f. We assume that $u^z(\cdot)$ is *increasing*, *convex* and *differentiable* in v for all z. We impose no restrictions on how u depends on z. The buyer's payoff from not obtaining any objects is normalized to *zero*.

By the revelation principle we know that the seller can without loss of generality restrict attention to incentive compatible direct revelation mechanisms.

A *direct revelation mechanism, (DRM)*, $M = (p, x)$ consists of an *assignment rule* $p : V \longrightarrow \Delta(Z)$ and a *payment rule* $x : V \longrightarrow \mathbb{R}$. Given a report v, the assignment rule specifies the probability of each allocation and the payment rule specifies the expected payment. We denote by $p^z(v)$ the probability that allocation z is implemented when the report is v.

The interim expected utility of type v buyer when he participates and declares v' is

$$U(v, v'; (p, x)) = \sum_{z \in Z} p^z(v') u^z(v) - x(v').$$

Definition 1. We say that a mechanism (p, x) is *feasible* iff for all $v, v' \in V$ it satisfies

(IC) **"incentive constraints,"** $U(v, v; (p, x)) \geq U(v, v'; (p, x))$
(PC) **"participation constraints"** $U(v, v; (p, x)) \geq 0$
(RES) **"resource constraints"**[2] $\sum_{z \in Z} p^z(v) = 1$, $p^z(v) \geq 0$

[2]Notice that Z contains the allocation where the seller keeps all the objects, thus $\sum_{z \in Z} p^z(c) = 1$.

With the help of the revelation principle the seller's problem can be written as

$$\max_{(p,x)} \int_V x(v)f(v)dv \tag{1}$$

subject to (p,x) being "feasible."

Given a *DRM* (p,x) a buyer's maximized payoff is given by

$$U(v) \equiv \max_{v'} \sum_{z\in Z} p^z(v')u^z(v) - x(v'),$$

and it is convex, since it is a maximum of convex functions. It is then easy to prove that its derivative

$$P(v) \equiv \sum_{z\in Z} p^z(v)\frac{\partial u^z(v)}{\partial v} \tag{2}$$

(more precisely a selection from its subgradient, which is single valued almost surely) is increasing. Necessary and sufficient conditions for incentive compatibility can then be obtained with relatively standard arguments, (see for instance Figueroa and Skreta (2006a)), and are stated in the following Lemma.

Lemma 1 *A mechanism (p,x) is incentive compatible iff*

$$\begin{aligned} P(v') &\geq P(v) && \textit{for all } v' > v \\ U(v) &= U(\underline{v}) + \int_{\underline{v}}^{v} P(s)ds && \textit{for all } v \in V. \end{aligned} \tag{3}$$

Then, letting

$$J_z(v) \equiv u^z(v) - \frac{[1-F(v)]}{f(v)}\frac{\partial u^z(v)}{\partial v}$$

denote the virtual utility of allocation z, and using standard arguments, we can rewrite the seller's problem as in the next lemma

Lemma 2 *The seller's problem can be reduced to find an allocation rule p that solves*

$$\max_{(p,x)} \int_V \sum_{z\in Z} p^z(v)J_z(v)f(v)dv \tag{4}$$

$$s.t.\ P(v) \textit{ increasing in } v$$

$$\sum_{z\in Z} p^z(v) = 1 \textit{ and } p^z(v) \geq 0.$$

In the next section we proceed with the analysis of this problem.

3. Analysis of the Problem

3.1 Incentive Constraints May Bind Despite Monotone Virtual Utilities

The solution of the program stated in (4) is straightforward if the assignment rule obtained by pointwise maximization of $\int_V \sum_{z \in Z} p^z(v) J_z(v) f(v) dv$ also satisfies the requirements of P being increasing. Following Myerson (1981) we will refer to this as the *regular* case. On the other hand, in the *general* case, pointwise optimization will lead to a mechanism that may not be feasible.

In Myerson (1981), pointwise optimization will lead to a feasible solution if the virtual utility is increasing in a buyer's valuation.[3] For the cases where the virtual utility is not increasing, Myerson (1981) presents a clever rewriting of the objective function in terms of "ironed" virtual utilities. He shows that pointwise maximization of this artificial objective function leads to a feasible solution and moreover the solution of this artificial program solves the original one as well. Unfortunately, this technique does not work here, because as we will now argue, incentive compatibility may be violated even if virtual utilities are strictly increasing and thus "ironed".

Pointwise optimization assigns probability one to the allocation with the highest virtual utility at each vector of types. Recalling that IC requires P to be increasing in v we notice that along a region of types where the same allocation z is selected throughout, $P(v) = \frac{\partial u^z(v)}{\partial v} \equiv P^z(v)$ is increasing by the convexity of $u^z(\cdot)$. Incentive compatibility can be violated though when the seller wishes to switch, say, from allocation z_1 to z_2. At such a point, call it v^*, IC requires that $P^{z_2}(v^*) \geq P^{z_1}(v^*)$, however this condition may fail because, as it is depicted in Figure 1, it is possible that $P^{z_2}(v^*) < P^{z_1}(v^*)$. We call P^z the *"marginal impact of type on allocation z."*

As illustrated in Figure 1, an assignment rule obtained via pointwise optimization may fail to satisfy incentive compatibility constraints, even if the virtual utilities of all allocations are *strictly monotonic* in a buyer's type.

Corollary 3 *Incentive compatibility may bind at a solution even if the virtual utilities of all allocations are strictly monotonic in the buyers type.*

This Corollary is in contrast to one of the standard lesson from the theory of revenue maximizing mechanisms. In Figueroa and Skreta (2006) we state sufficient conditions for pointwise optimization to lead to an incentive compatible allocation rule. In this paper our

[3] A sufficient condition for this is that F satisfies the monotone hazard rate property, (MHR).

objective is to illustrate how one can solve for a revenue maximizing auction when such conditions are violated and show its implications for the shape of the optimal mechanism.

Before providing a specific example where this phenomenon occurs, it may be worth highlighting the features of the classical environment that guarantee that increasing virtual valuations imply that pointwise optimization will lead to a feasible solution.

In the standard problem, from each buyer's perspective there are two allocations: either he obtains the good or not. In the first case, he gets a payoff of $u_i^{z_i}(v_i) = v_i$, and in the latter, a payoff of $u_i^{z_j}(v_i) = 0$. Here z_j stands for j getting the good, with $j \neq i$ and j could also be the seller. The $P^{z\prime}s$ is this case are:

$$\begin{array}{r} P^{z_i}(v_i) = \frac{\partial u_i^{z_i}(v_i)}{\partial v_i} = 1 \\ P^{z_j}(v_i) = \frac{\partial u_i^{z_j}(v_i)}{\partial v_i} = 0 \text{ for all } j \neq i \end{array}.$$

The fact that there is only one relevant allocation from $i's$ perspective implies if the virtual utility of that allocation, namely, $J_{z_i}(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$, is increasing in v_i, then the allocation rule obtained via pointwise optimization will be incentive compatible. This can be seen immediately from Figure 2.

Now we present a specific example where incentive constraints bind despite monotone virtual utilities.

Example : Incentive Compatibility may Bind Despite Strictly Increasing Virtual utilities

Suppose that there is a single buyer whose preference parameter, v, is distributed uniformly on the interval $[0, 1]$ and that there are two possible allocations, z_1 and z_2. For each preference parameter realization the payoffs arising from these two allocations are given by

$$\begin{aligned} u_{z_1}(v) &= 0.5e^v + 0.524 \\ u_{z_2}(v) &= e^{0.5v}, \end{aligned}$$

which are both convex and increasing in v. The virtual utilities of allocations z_1 and z_2 are respectively given by

$$\begin{aligned} J_{z_1}(v) &= 0.5ve^v + 0.524 \\ J_{z_2}(v) &\equiv e^{0.5v}(0.5 + 0.5v) \end{aligned}$$

and are strictly increasing in v. They are depicted in Figure 3.

However, in this example despite the fact that the virtual utilities of both allocations z_1 and z_2 are strictly increasing in v, pointwise optimization of

$$\max_{(p,x)} \int_V \left[p^{z_1}(v) J_{z_1}(v) + p^{z_2}(v) J_{z_2}(v) \right] f(v) dv$$

does not lead to a feasible mechanism. As we can see from Figure 3, pointwise optimization dictates to assign probability one to allocation z_1 for $v < 0.1054$ and to assign probability one to allocation z_2 for v in the interval $[0.1054, 0.6346]$. However it is not possible to switch from allocation z_1 to allocation z_2 because this would imply that P falls at 0.1054, since as can be seen in Figure 4, $P^{z_1}(v) > P^{z_2}(v)$ for all $v \in (0, 1]$.

In the following section we describe how one can proceed to solve for an optimal mechanism in cases where incentive constraints bind, and show that in this case the optimal mechanism can be random.

3.2 A Simple Case: Bunching with Two Allocations

We will illustrate how one can solve for an optimal mechanism in a very simple scenario. This has the advantage that it allows us to avoid tedious case-specific details.

Assume that there are only two allocations that the seller can choose from, z_1 and z_2. Suppose also for simplicity that the *"marginal impact of type"* is always higher for z_1 than z_2 for all interior v, that is

$$P^{z_1}(v) > P^{z_2}(v),$$

which is equivalent to

$$\frac{\partial u^{z_1}(v)}{\partial v} > \frac{\partial u^{z_2}(v)}{\partial v}.$$

Let J_{z_1} denote the virtual utility of allocation z_1 and J_{z_2} denote the virtual utility of allocation z_2. Furthermore, suppose that there is *only one* point, call it v^*, where the seller wishes to switch from allocation z_1 to allocation z_2. Such a situation is depicted in Figure 1. In such a scenario, the assignment rule obtained via pointwise optimization would violate IC at v^*. How will the seller proceed? At a solution the seller should mix between z_1 and z_2 in a way that minimizes the "cost"of having to choose with positive probability an allocation that does not have the highest virtual utility, subject to respecting the incentive compatibility constraints.

The region of "compromise"is an interval of the form $[\underline{x}, \overline{x}]$, where that satisfies the inequalities $\underline{v} \leq \underline{x} \leq v^* \leq \overline{x} \leq \hat{v}$, where $\hat{v}$ is the first point to the right of v^* where J_{z_1} and

J_{z_2} cross again.[4] The loss of assigning positive weight to allocation z_2 for $v \in [\underline{x}, v^*)$ is given by $\int_{\underline{x}}^{v^*} p^{z_2}(v) \left[J_{z_1}(v) - J_{z_2}(v)\right] f(v)dv$ and the loss of assigning positive weight to allocation z_2 for $v \in [\underline{x}, v^*)$ is given by $\int_{v^*}^{\bar{x}} p^{z_1}(v) \left[J_{z_2}(v) - J_{z_1}(v)\right] f(v)dv$. An optimal mechanism must randomize between z_1 and z_2 on $[\underline{x}, \overline{x}]$, in a way such that the loss is minimized. Moreover $\underline{x}$ and $\bar{x}$ must be chosen optimally.

The problem to be solved is called Program A and it is given by[5]:

$$\min_{p^{z_1}, \underline{x}, \overline{x}} \int_{\underline{x}}^{v^*} (1 - p^{z_1}(v)) \left[J_{z_1}(v) - J_{z_2}(v)\right] f(v)dv + \int_{v^*}^{\bar{x}} p^{z_1}(v) \left[J_{z_2}(v) - J_{z_1}(v)\right] f(v)dv.$$

subject to:

(i) $P(v) \equiv p^{z_1}(v)\frac{\partial u^{z_1}(v)}{\partial v} + (1 - p^{z_1}(v))\frac{\partial u^{z_2}(v)}{\partial v}$ increasing in v for $v \in [\underline{x}, \overline{x}]$ and

(ii) $P(v) \geq P(\underline{x})$

(iii) $\underline{v} \leq \underline{x} \leq v^* \leq \overline{x} \leq \hat{v}$

The constraints (i) and (ii) are the incentive compatibility constraints.

Now we will establish that Program A is equivalent to a much simpler problem where the only choice variable is $\underline{x}$. This will be done with the help of a few auxiliary results that we demonstrate next.

Our first result states that an optimal assignment rule randomizes between allocation z_1 and allocation z_2 in such a way, such that P remains constant over $[\underline{x}, \overline{x}]$.

Lemma 4 *An optimal assignment rule randomizes between allocations z_1 and z_2 over an interval $[\underline{x}, \overline{x}]$ with $\underline{v} \leq \underline{x} \leq \overline{x} \leq \hat{v}$ in a way such that*

$$P(v) \equiv p^{z_1}(\underline{x})\frac{\partial u^{z_1}(\underline{x})}{\partial v} + (1 - p^{z_1}(\underline{x}))\frac{\partial u^{z_2}(\underline{x})}{\partial v} \equiv P(\underline{x}). \tag{5}$$

Proof. We argue by contradiction. First suppose that P were increasing in $[v^*, \overline{x}]$. This cannot be optimal because the seller can assign more weight to allocation z_2, up to the point where P becomes flat. This will increase revenue because in the interval $(v^*, \overline{x}]$ z_2 is preferred to z_1. Now if P were increasing in $[\underline{x}, v^*)$, the seller can increase revenue by assigning more weight to allocation z_1 for $v < v^*$ up to a point where P becomes flat and reaches the level of $P(v^*)$. Hence at an optimum P must be flat on $[\underline{x}, \bar{x}]$. ∎

[4]Such a point exists, since $J_{z_i}(\overline{v}) = u^{z_i}(v)$, and we have that $\frac{du^{z_1}(v)}{dv} > \frac{du^{z_2}(v)}{dv}$, so $J_{z_1}(\overline{v}) > J_{z_2}(\overline{v})$ unless $u^{z_2}(\underline{v}) > u^{z_1}(\underline{v})$. This last situation is not possible since then we would not have the first crossing.

[5]Since for each v it must be that $p^{z_1}(v) + p^{z_2}(v) = 1$, we will express $p^{z_2}(v)$ as $1 - p^{z_1}(v)$.

The intuition for this result is simple. Optimality dictates that P is as small as possible to the right of v^*, where z_2 is preferred by the seller, and P is as large as possible to the left of v^*, where the seller prefers z_1. Since P must be increasing these two forces imply that for the interval where the seller is mixing P must be flat. Put in another way, since the incentive constraint is binding, optimality dictates that it is satisfied with "equality", so $P(\cdot)$ is flat and not strictly increasing.

Our next result establishes that at an optimal allocation rule it must be the case that if $\underline{x} > \underline{v}$, then the seller at $\underline{x}$ assigns probability one at z_1, whereas if $\underline{x} = \underline{v}$, then she assigns probability one to allocation z_2 at that point.

Lemma 5 *If at an optimum* $\underline{x} > \underline{v} <$ *then* $p^{z_1}(\underline{x}) = 1,$. *If at an optimum* $\underline{x} = \underline{v}$, *then* $p^{z_1}(\underline{x}) = 0$.

Proof. See Appendix. ■

Now with the help of Lemmata 4 and 5 we can obtain an expression for the optimal mixing merely as a function of $\underline{x}$. In particular if $\underline{x} > \underline{v}$, Lemma 5 implies that $p^{z_1}(\underline{x}) = 1$ and with the help of (5) we get

$$p^{z_1}(v)\frac{\partial u^{z_1}(v)}{\partial v} + (1 - p^{z_1}(v))\frac{\partial u^{z_2}(v)}{\partial v} = \frac{\partial u^{z_1}(\underline{x})}{\partial v}$$

which implies that at a solution

$$p^{z_1}(v) = \frac{\frac{\partial u^{z_1}(\underline{x})}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}}{\frac{\partial u^{z_1}(v)}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}}, \text{ when } \underline{x} > \underline{v}. \tag{6}$$

Now (6) allows us also to find $\bar{x}$; it is either the smallest v where $p^{z_1}(v) = 0$, or if such a v does not exist, it is equal to $\bar{v}$, that is

$$\overline{x} = \min\{\hat{v}, \min\{v \in [\underline{x}, \hat{v}] | p^{z_1}(v) = 0\}\}. \tag{7}$$

Now, if $\underline{x} = \underline{v}$, Lemma 5 tells us that $p^{z_1}(\underline{v}) = 0$, which is equivalent to $p^{z_2}(\underline{v}) = 1$, which immediately implies that $\underline{v} = \underline{x} = \bar{x}$ and therefore

$$p^{z_2}(v) = 1 \text{ for all } v \in [\underline{v}, \bar{v}], \tag{8}$$

and

From (6)-(8) one can see that Program A can be stated as a problem where the control variable is simply $\underline{x}$, which we denote by Program B:

$$\min_{\underline{x}\in[\underline{v},v^*]} \int_{\underline{x}}^{v^*} (1-p^{z_1}(\underline{x})(v))\,[J_{z_1}(v) - J_{z_2}(v)]\, f(v)dv + \int_{v^*}^{\bar{x}(\underline{x})} p^{z_1}(\underline{x})(v)\,[J_{z_2}(v) - J_{z_1}(v)]\, f(v)dv, \tag{9}$$

where $p^{z_1}(\underline{x})$ and $\bar{x}(\underline{x})$ are given by (6)-(8).

An optimal allocation rule is then of the form,

$$\begin{aligned} p^*(v) &= (p^{z_1}(v), p^{z_2}(v)) = (1,0) \text{ for } v \in [\underline{v}, \underline{x}) \qquad (10)\\ p^*(v) &= \left(\frac{\frac{\partial u^{z_1}(\underline{x})}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}}{\frac{\partial u^{z_1}(v)}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}}, 1 - \frac{\frac{\partial u^{z_1}(\underline{x})}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}}{\frac{\partial u^{z_1}(v)}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}} \right) \text{ for } [\underline{x}, \overline{x}] \\ p^*(v) &= (p^{z_1}(v), p^{z_2}(v)) = (0,1) \text{ for } v \in (\overline{x}, \hat{v}], \end{aligned}$$

and where $\underline{x}$ solves Program B.

If at a solution of Program B we have that $\underline{x} = \underline{v}$, then p^* is

$$p^*(v) = (0,1) \text{ for } v \in [\underline{v}, \hat{v}].$$

The payment rule can be obtained from the allocation rule exactly as in Myerson (1981). We will now solve Program B for our Example.

Optimality of Randomizations:Illustration

For our example[6], (9) becomes:

$$\begin{aligned} &\int_{\underline{x}}^{0.1054} \frac{e^{\underline{x}} - e^{v}}{e^{0.5v} - e^{v}} \left[0.5ve^{v} - e^{0.5v}(0.5v+0.5) + 0.524\right] dv \\ &+ \int_{0.1054}^{2\underline{x}} \frac{e^{0.5v} - e^{\underline{x}}}{e^{0.5v} - e^{v}} \left[e^{0.5v}(0.5v+0.5) - 0.5ve^{v} - 0.524\right] dv. \end{aligned}$$

This function is depicted in Figure 5.

It has a unique minimizer at

$$\underline{x} = 0.074, \tag{11}$$

which with the help of (6) and (7) implies that[7]

$$\begin{aligned} p^{z_1}(v) &= \frac{e^{0.5v} - e^{0.074}}{e^{0.5v} - e^{v}} \\ \bar{x} &= 0.148. \end{aligned}$$

[6]Strictly speaking, our example does not fit all the specifications of the simple case we analyzed, because J_{z_1} and J_{z_2} cross twice.

[7]Calculations and graphs for this example have been done with Matlab. Code available upon request.

The optimal assignment rule for this example is given by:

$$
\begin{aligned}
p^*(v) &= (1,0) \text{ for } v \in [0, 0.074) \\
p^*(v) &= (\frac{e^{0.5v} - e^{0.074}}{e^{0.5v} - e^{v}}, 1 - \frac{e^{0.5v} - e^{0.074}}{e^{0.5v} - e^{v}}) \text{ for } v \in [0.074, 0.148) \\
p^*(v) &= (0,1) \text{ for } v \in [0.148, 0.6346] \\
p^*(v) &= (1,0) \text{ for } v \in (0.6346, 1].
\end{aligned} \tag{12}
$$

With the help of Figure 4, one can easily see that p^* in (12) is incentive compatible. Figure 6 depicts the probability of z_1 that p^* assigns around the region of randomization.

In this example the optimal assignment rule involves randomizations. This in contrast to the classical case, where, (excluding cases where the seller is indifferent), an optimal allocation rule never involves randomizations.

4. Concluding Remarks

We illustrated how one can solve for an optimal mechanism in a very simple scenario and noted this has the advantage that it allows us to avoid tedious case-specific details. In general, when there are more allocations, and/or when virtual utilities cross more than once, and/or when $P^{z\prime}s$ are not always ranked in the same way, the details of the solution will depend on the particular specifics of the problem at hand. However, the main idea of how to proceed is the one we illustrated. Whenever there is a point where IC is violated by the assignment rule obtained via pointwise optimization, a solution will involve an interval of randomization between more than one allocations. Of course, it is possible that in some cases this interval will be degenerate. Finally, it is worth mentioning that our technique can be also useful in cases where the designer is interested in designing efficient mechanisms, and where the allocation rule that maximizes social welfare is not incentive compatible. In these cases our method can be used to find the allocation that gets as close as possible to maximizing social welfare, subject to the incentive constraints.

5. Appendix

Proof of Lemma 5

Recall from (5) that all $v \in [\underline{x}, \overline{x}]$ we have that

$$p^{z_1}(v)\frac{\partial u^{z_1}(v)}{\partial v} + (1 - p^{z_1}(v))\frac{\partial u^{z_2}(v)}{\partial v} = \left[p^{z_1}(v)\frac{\partial u^{z_1}(v)}{\partial v} + (1 - p^{z_1}(v))\frac{\partial u^{z_2}(v)}{\partial v}\right]\Bigg|_{\underline{x}}. \quad (13)$$

Now from (13) we can obtain that

$$p^{z_1}(v) = \frac{p^{z_1}(\underline{x}) \left.\frac{\partial u^{z_1}(v)}{\partial v}\right|_{\underline{x}} + (1 - p^{z_1}(\underline{x})) \left.\frac{\partial u^{z_2}(v)}{\partial v}\right|_{\underline{x}} - \frac{\partial u^{z_2}(v)}{\partial v}}{\frac{\partial u^{z_1}(v)}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}}$$

so the objective function can be written as

$$\begin{aligned} R(\underline{x}, p^{z_1}(\underline{x})) &= \int_{\underline{x}}^{v^*} \frac{-p^{z_1}(\underline{x}) \left.\frac{\partial u^{z_1}(v)}{\partial v}\right|_{\underline{x}} - (1 - p^{z_1}(\underline{x})) \left.\frac{\partial u^{z_2}(v)}{\partial v}\right|_{\underline{x}} - \frac{\partial u^{z_2}(v)}{\partial v}}{\frac{\partial u^{z_1}(v)}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}} \left[J_{z_1}(v) - J_{z_2}(v)\right] f(v)dv \\ &+ \int_{v^*}^{\overline{x}(\underline{x}, p^{z_1}(\underline{x}))} \frac{p^{z_1}(\underline{x}) \left.\frac{\partial u^{z_1}(v)}{\partial v}\right|_{\underline{x}} + (1 - p^{z_1}(\underline{x})) \left.\frac{\partial u^{z_2}(v)}{\partial v}\right|_{\underline{x}} - \frac{\partial u^{z_2}(v)}{\partial v}}{\frac{\partial u^{z_1}(v)}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}} \left[J_{z_2}(v) - J_{z_1}(v)\right] f(v)dv \end{aligned}$$

It is easy to see that, because of feasibility, if $\underline{x} > \underline{v}$, then $p^{z_1}(\underline{x}) = 1$. We can divide then in two regions:

$$R(\underline{x}, p^{z_1}(\underline{x})) = \begin{cases} R(\underline{x}, 1) & \text{if } \underline{x} > \underline{v} \\ R(\underline{v}, p^{z_1}(\underline{v})) & \text{if } \underline{x} = \underline{v} \end{cases}$$

To find the minimum of R, we look separately in each region. First, we constraint ourselves to look for in the region for which $p^{z_1}(\underline{x}) = 1$. Differentiating R with respect to $\underline{x}$ we get

$$\begin{aligned} \frac{dR(\underline{x})}{d\underline{x}} &= -\left[\frac{\frac{\partial u^{z_1}(v)}{\partial v} - p^{z_1}(v)\frac{\partial u^{z_1}(v)}{\partial v} - (1 - p^{z_1}(v))\frac{\partial u_(v)}{\partial v}}{\frac{\partial u^{z_1}(v)}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}}\right]\Bigg|_{\underline{x}} \left[J_{z_1}(\underline{x}) - J_{z_2}(\underline{x})\right] f(\underline{x}) \\ &+ \left[\frac{p^{z_1}(\underline{x}) \left.\frac{\partial u^{z_1}(v)}{\partial v}\right|_{\underline{x}} + (1 - p^{z_1}(\underline{x})) \left.\frac{\partial u^{z_2}(v)}{\partial v}\right|_{\underline{x}} - \frac{\partial u^{z_2}}{\partial v}}{\frac{\partial u^{z_1}(v)}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}}\right]\Bigg|_{\overline{x}(\underline{x})} \left[J_{z_2}(\overline{x}(\underline{x})) - J_{z_1}(\overline{x}(\underline{x}))\right] \frac{d\overline{x}(\underline{x})}{d\underline{x}} f(\overline{x}(\underline{x})) \\ &+ \int_{\underline{x}}^{\overline{x}(\underline{x})} \frac{p^{z_1}(\underline{x}) \left.\frac{\partial^2 u^{z_1}(v)}{\partial v^2}\right|_{\underline{x}} + (1 - p^{z_1}(\underline{x})) \left.\frac{\partial^2 u^{z_2}(v)}{\partial v^2}\right|_{\underline{x}}}{\frac{\partial u^{z_1}(v)}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}} \left[J_{z_2}(v) - J_{z_1}(v)\right] f(v)dv \end{aligned}$$

Notice also that the second term vanishes since, for a given $\underline{x}$ either $\overline{x}(\underline{x}) = \hat{v}$ and therefore $\frac{d\overline{x}(\underline{x})}{d\underline{x}} = 0$, or

$$p^{z_1}(\overline{x}(\underline{x})) \equiv \left[\frac{p^{z_1}(\underline{x}) \left.\frac{\partial u^{z_1}(v)}{\partial v}\right|_{\underline{x}} + (1 - p^{z_1}(\underline{x})) \left.\frac{\partial u^{z_2}(v)}{\partial v}\right|_{\underline{x}} - \frac{\partial u^{z_2}}{\partial v}}{\frac{\partial u^{z_1}(v)}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}} \right]\Bigg|_{\overline{x}(\underline{x})} [J_{z_2}(\overline{x}(\underline{x})) - J_{z_1}(\overline{x}(\underline{x}))] \frac{d\overline{x}(\underline{x})}{d\underline{x}} f(\overline{x}(\underline{x})) = 0$$

We can then write:

$$\frac{dR(\underline{x})}{d\underline{x}} = \left.\frac{d^2 u^{z_1}(v)}{dv^2}\right|_{\underline{x}} \int_{\underline{x}}^{\overline{x}(\underline{x})} \frac{J_{z_2}(v) - J_{z_1}(v)}{\frac{du^{z_1}(v)}{dv} - \frac{du^{z_2}(v)}{dv}} f(v)dv$$

It's easy to see that the sign of $\frac{dR(\underline{x})}{d\underline{x}}$ depends only on the sign of $\int_{\underline{x}}^{\overline{x}(\underline{x})} \frac{J_{z_2}(v) - J_{z_1}(v)}{\frac{\partial u^{z_1}(v)}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}} f(v)dv$, since $\left.\frac{d^2 u^{z_1}(v)}{dv^2}\right|_{\underline{x}}$ is positive by convexity.

Moreover, $\int_{\underline{x}}^{\overline{x}(\underline{x})} \frac{J_{z_2}(v) - J_{z_1}(v)}{\frac{\partial u^{z_1}(v)}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}} f(v)dv$ is nondecreasing in $\underline{x}$: the integrand is negative when $v < v^*$ and positive otherwise, and $\overline{x}(\underline{x})$ is nondecreasing in $\underline{x}$.

This observation, plus the fact that $\left[\frac{\partial u^{z_1}(v)}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}\right]\Big|_{\underline{x}=v^*} > 0$ and $\int_{v^*}^{\overline{x}(\underline{v})} \frac{J_{z_2}(v) - J_{z_1}(v)}{\frac{\partial u^{z_1}(v)}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}} f(v)dv >$ 0 implies that $\frac{dR(\underline{x})}{d\underline{x}}$ is either always positive, or that it crosses 0 at some point. This gives us:

$$\underline{x} = \begin{cases} \underline{v} & \text{if } \int_{\underline{v}}^{\overline{x}(\underline{v})} \frac{J_{z_2}(v) - J_{z_1}(v)}{\frac{\partial u^{z_1}(v)}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}} f(v)dv \geq 0 \\ \left[\frac{dR(\underline{x})}{d\underline{x}}\right]^{-1}(0) & \text{if not} \end{cases} \tag{14}$$

This equations gives us the optimal starting point when we are constrained to an optimal starting mixture of $p^{z_1}(\underline{x}) = 1$. We have to analyze now the case when $p^{z_1}(\underline{x}) \neq 1$. We have already established that this can only happen when $\underline{x} = \underline{v}$. Doing similar manipulations we get

$$\frac{dR(p^{z_1}(\underline{v}))}{dp^{z_1}(\underline{v})} = \left[\frac{\partial u^{z_1}(v)}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}\right]\Bigg|_{\underline{x}} \int_{\underline{x}}^{\overline{x}(\underline{x})} \frac{J_{z_2}(v) - J_{z_1}(v)}{\frac{\partial u^{z_1}(v)}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}} f(v)dv$$

From which we conclude, using analogous arguments, that

$$p^{z_1}(\underline{v}) = \begin{cases} 0 & \text{if } \int\limits_{\underline{v}}^{\overline{x}(\underline{v})} \frac{J_{z_2}(v) - J_{z_1}(v)}{\frac{\partial u^{z_1}(v)}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}} f(v)dv \geq 0 \\ 1 & \text{if not} \end{cases} \tag{15}$$

Therefore, by (14) and (15) everything depends on the sign of $\int\limits_{\underline{v}}^{\overline{x}(\underline{v})} \frac{J_{z_2}(v) - J_{z_1}(v)}{\frac{\partial u^{z_1}(v)}{\partial v} - \frac{\partial u^{z_2}(v)}{\partial v}} f(v)dv$. If positive then $\underline{x} = \underline{v}$ dominates any other starting point, and moreover $p^{z_1}(\underline{v}) = 0$. If negative, then the optimal $\underline{x}$ is interior, and we know that $p^{z_1}(\underline{v}) = 1$.

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